STABILITY OF A PROCESSOR SHARING QUEUE WITH VARYING THROUGHPUT

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Abstract. In this paper, we present a stability criterion for Processor Sharing queues, in which the throughput may depend on the number of customers in the system (in such cases such as interferences between the users). Such a system is represented by a point measure-valued stochastic recursion keeping track of the remaining processing times of the customers.

1. Introduction

In this paper, we address the question of stationarity in the general ergodic framework for processor sharing queues, in which the throughput (i.e. the quantity of work achieved by the server(s) per unit of time) may depend on the state of system. More precisely, we assume hereafter that the server(s) (it will be clear in the sequel that the effective number of servers does not really matter, only does the quantity of work consumed per unit of time) process(es) all the jobs present in the system simultaneously and fairly. Whenever there are n customers in the system, each of them is thus served at a rate that depend on n, say $r(n)$. The classical case is when $r(n) = 1/n$, $n \geq 1$, so that the total throughput equals $n.r(n) = 1$ whenever the system is non-empty: this is the classical Processor Sharing queue. Hereafter we consider a more general context, in which the total throughput may decrease with the number of customers in the system (hence $n.r(n) \leq 1$). This is the case for instance in a wireless network in which the number of users being currently active may decrease the efficiency of the resources. Another case, is when the value of n the number of customers does *not* change the nominal service rate $r(n)$, say $r(n) = 1$ for all n. This corresponds to the classical queue with infinitely many servers.

In both cases and under general stationary ergodic assumptions, Loynes' stability result does not hold since this is not a proper G/G/1 queue (the throughput may be less, or larger than one). We address the question of the existence of a stationary version of such queues by representing them with point measure-valued stochastic recursions in the Palm setting, so as to take into account the dependency in the number of customers. This point measures keep track of all the remaining service times of all the customers in the system. Then, it is possible to provide conditions for the existence of a stationary version of this sequence, that allow to explicitly construct stationary queues under these assumptions.

This paper is organized as follows. After some preliminaries in section 2, we present the queueing models we consider in section 3. In section 4 we study the

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particular case of the G/G/∞ queue, and in section 5 we present a stability criterion for generalized processor queues with state-dependent throughput.

2. Preliminaries

Let $\mathbf{M}_f^+$ and $\mathcal{C}_b$ denote respectively the set of positive finite measures on $\mathbb{R}_+^*$ and the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Equipped with the *weak topology* $\sigma\left(\mathbf{M}_f^+, \mathcal{C}_b\right)$, $\mathbf{M}_f^+$ is Polish (see [2]). Let $\mathbf{0}$ be the zero measure on $\mathbb{R}$ (i.e., such that $\mathbf{0}(\mathfrak{B}) = 0$ for any Borel set $\mathfrak{B}$ on $\mathbb{R}$). For any $\mu \in \mathbf{M}_f^+$ and any measurable $f\ :\mathbb{R} \to \mathbb{R}$, we classically write $\langle \mu, f\rangle := \int f\, d\mu$. Let us denote for any $y \in \mathbb{R}$ and any measurable $f : \mathbb{R} \to \mathbb{R}$, $\tau_y f(.) = f(.-y)\mathbf{1}_{\{.>y\}}$. Then, for any $\mu \in \mathbf{M}_f^+$, $\tau_y\mu$ denotes the only element of $\mathbf{M}_f^+$ s.t. $\langle \tau_y\mu, f\rangle = \langle \mu, \tau_y f\rangle$.

Let the set $\mathbf{M}_f^+$ be endowed with the *increasing partial integral order* $\preceq$: for any two $\mu, \nu \in \mathbf{M}_f^+$, $\mu \preceq \nu$ if $\langle \mu, f\rangle \leq \langle \nu, f\rangle$ for any measurable non-decreasing function f such that these integrals exist. Of course, $\mathbf{0} \preceq \mu$ for any $\mu \in \mathbf{M}_f^+$. Furthermore, let us remark that

Lemma 1. *Any sequence of $\mathbf{M}_f^+$ that is $\preceq$-increasing and bounded above converges for the weak topology.*

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a $\preceq$-increasing sequence of $\mathbf{M}_f^+$ that is bounded above by $\mu \in \mathbf{M}_f^+$. Then, as easily seen the sequence of non-increasing real functions $\{\mu_n\left([.,\infty)\right)\}_{n\in\mathbb{N}}$ tends pointwise, and hence (this is Diniz Theorem), uniformly to a non-increasing real function f that is right continuous and has a countable number of discontinuities. Moreover $f(0) \leq \mu(\mathbb{R}_+^*) < \infty$, and we can fully characterize a measure $\mu^* \in \mathbf{M}_f^+$ setting $\mu^*\left((0,x)\right) = f(0) - f(x)$ for all $x \in \mathbb{R}_+^*$. In particular, $\sup_{x\in\mathbb{R}_+^*} \left|\mu^n\left((0,x)\right) - \mu^*\left((0,x)\right)\right| \underset{n\to\infty}{\longrightarrow} 0$, hence μ^n tends to μ^* in total variation, which completes the proof. $\square$

Let now $\mathcal{M} \subset \mathbf{M}_f^+$ be the subset of finite (simple) counting measures on $\mathbb{R}_+^*$. Any $\mu \in \mathcal{M}\backslash\{\mathbf{0}\}$ reads $\mu = \sum_{i=1}^{N(\mu)} \delta_{\alpha_i(\mu)}$, where $N(\mu) := \mu(\mathbb{R}_+^*)$ is the number of atoms of μ, δ_x is the Dirac measure at $x \in \mathbb{R}+$ and $\alpha_1(\mu) < \alpha_2(\mu) < ... < \alpha_{N(\mu)}(\mu)$. Then, $\tau_y(\mu) = \sum_{i=1}^{N(\mu)} \delta_{\alpha_i(\mu)-y}\mathbf{1}_{\{\alpha_i(\mu)>y\}}$ and for any two $\mu, \nu \in \mathcal{M}\backslash\{\mathbf{0}\}$, $\mu \preceq \nu$ whenever

$$\begin{cases} (i) \quad N(\mu) \leq N(\nu), \\ \\ (ii) \quad \text{for all } i = 0, ..., N(\mu)-1, \alpha_{N(\mu)-i}(\mu) \leq \alpha_{N(\nu)-i}(\nu). \end{cases}$$

We denote for any $\mu \in \mathcal{M}\backslash\{\mathbf{0}\}$, $Z(\mu) = \alpha_{N(\mu)}(\mu)$, the largest atom of μ. Finally, we write $x^+ = \max(x, 0)$ for any real number x, $\sum_{i=j}^k . \equiv 0$ whenever $k < j$ and $\max\{\emptyset\} \equiv 0$.

3. The model

Let us first introduce our definitions and assumptions on the queueing systems we shall consider in the sequel. Let $(\Omega, \mathcal{F}, \mathbf{P}, \theta_t)$ be a probability space furnished with a bijective flow $(\theta_t)_{t\geq 0}$, under which $\mathbf{P}$ is stationary and ergodic. Define on Ω the θ_t-compatible simple point process $(A_t)_{t\in\mathbb{R}}$ of points $... < T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < ...$, that represent the arrival times of the customers in a queue without buffer. The process $(A_t)_{t\in\mathbb{R}}$ is marked by a sequence $\{\sigma_n\}_{n\in\mathbb{Z}}$, where for

all $n \in \mathbb{Z}$, σ_n is the service duration requested by the customer C_n arrived at time T_n. Also denote for all $n \in \mathbb{Z}$, $\xi_n = T_{n+1} - T_n$, and suppose that the generic r.v. σ and ξ are integrable. We consider that the server(s) follow a generalized Processor Sharing discipline. By that, we mean that all present customers are taken care of simultaneously, at a rate r that is equal for all customers. An example is of course provided by the classical Processor Sharing queue, but it will be shown in the subsequent sections that significant results can be obtained as well for a wider class of systems. Indeed, it is plausible to assume in many cases, that the amount of work in the system might affect the throughput, considering for instance the working cost induced by the switching mechanism in the processor, or the interferences between the users of a wireless network. In both cases, it is then natural to assume that the rate r is a non-increasing function of the service profile, *i.e.* $\mu \preceq \nu$ implies $r(\mu) \geq r(\nu)$. Hereafter, for the sake of simplicity, we will restrict to the sub-case, where r is a non-increasing function of the number of customers in the system, although it should be clear that all the results below hold as well when r is function of the whole service profile. In other words, at any t, denoting $Q(t)$ the number of customers in the system at t, each customer is allocated a quantity of work $r(Q_t)$ per unit of time, that is such that $r(i) \geq r(j)$ for all $i, j \in \mathbb{N}^*$ such that $i \leq j$. Let us illustrate through a naive example the effect of a large number of customers on the throughput.

	Nominal service rate	**Troughput**
1 customer	1	1
2 customer	0.495	0.99
3 customer	0.3	0.9
...	...	...
100 customers	0.008	0.8

Provided that C_n is in the system at t, his *remaining processing time* at this instant is the time before his service completion. The *service profile* of the system at t is the $\mathcal{M}$-valued process keeping track of the remaining processing times of all the customers in the system at t:

$$\mu(t) = \sum_{i=1}^{Q(t)} \delta_{\alpha_i(\mu(t))}$$

where $\alpha_1(\mu(t)) \leq \alpha_2(\mu(t)) \leq ..... \leq \alpha_{Q(t)}(\mu(t))$ denote the remaining processing times of the customers in the system at t, ranked in decreasing order. Let $W(t)$ denote the workload at t. Then, the workload and the congestion processes can easily be recovered from the service profile process by writing for all t

$$\begin{aligned} Q(t) &= N(\mu(t)), \\ W(t) &= \langle \mu(t), I \rangle, \end{aligned}$$

where I is the identity function. The processes μ, Q and W have RCLL paths, and we denote for all t $\mu(t-) = \lim_{s \uparrow\uparrow t} \mu(s)$ (and accordingly, $Q(t-)$ and $W(t-)$). We denote for all $n \in \mathbb{N}$, $\mu_n = \mu(T_n-)$ (respectively $Q_n = Q(T_n-)$, $W_n = X(T_n-)$) the service profile (resp. congestion, workload) just before the arrival of customer C_n.

Let $\left(\Omega, \mathcal{F}, \mathbf{P}^0\right)$ be the Palm space of A, denote $\theta := \theta_{T_1}$, θ^{-1} his measurable inverse and for all $n \in \mathbb{Z}$, $\theta^n = \theta \circ \theta \circ ... \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ ... \circ \theta^{-1}$. Note, that $\mathbf{P}^0$ is stationary and ergodic under θ, *i.e.* for all $\mathfrak{A} \in \mathcal{F}$, $\mathbf{P}^0\left[\theta^{-1}\mathfrak{A}\right] = \mathbf{P}^0\left[\mathfrak{A}\right]$

and $\theta\mathfrak{A} = \mathfrak{A}$ implies $\mathbf{P}^0[\mathfrak{A}] = 0$ or 1, and that all θ-contracting event (such that $\mathbf{P}^0\left[\mathfrak{A}^c \cap \theta^{-1}\mathfrak{A}\right] = 0$) is θ-invariant. Denoting $\xi := \xi_0$ and $\sigma := \sigma_0$, we have for all $n \in \mathbb{Z}$, $\xi_n := \xi \circ \theta^n$ and $\sigma_n := \sigma \circ \theta^n$.

We say that the E-valued random sequence $\{X_n\}_{n\in\mathbb{N}}$ is a stochastically recursive sequence (SRS) whenever for some random mapping $\phi:\, E \to E$,

$$X_{n+1} = \phi \circ \theta^n \left(X_n\right),\, n \in \mathbb{N},\, \mathbf{P}^0 - \text{ a.s..}$$

For any E-valued r.v. Y, we denote $\left\{X_n^{[Y]}\right\}_{n\in\mathbb{N}}$ the SRS $\{X_n\}_{n\in\mathbb{N}}$ such that $X_0^{[Y]} = Y$, $\mathbf{P}^0$-a.s.. We follow the formalism of [1] and formulate the question of stationarity for the SRS $\{X_n\}_{n\in\mathbb{N}}$ in the following terms. There exists a *stationary version* of $\{X_n\}_{n\in\mathbb{N}}$ whenever for some Y and for all n, $X_n^{[Y]} = Y \circ \theta^n$, $\mathbf{P}^0$-a.s., or in other words, provided that the equation

$$Y \circ \theta = \phi(Y)$$

admits a solution that is a E-valued r.v.. We say that two sequences of r.v. $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ couple provided that

$$\mathbf{P}^0\left[\exists N(\omega), X_n(\omega) = Y_n(\omega) \text{ for all } n \geq N(\omega)\right] = 1,$$

and that there is strong backwards from $\{X_n\}_{n\in\mathbb{N}}$ with the stationary sequence $\{Y \circ \theta^n\}$ whenever

$$\mathbf{P}^0\left[\exists N'(\omega), X_n \circ \theta^{-n}(\omega) = Y(\omega) \text{ for all } n \geq N'(\omega)\right] = 1.$$

Lemma 2. *The sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is stochastically recursive for any rate function r: letting for all $\mu \in \mathcal{M}$ and $x \in \mathbb{R}_+^*$,*

- *For all $i \leq N(\mu)$,*

$$\gamma_i^r(\mu, x) = r(N(\mu) - i + 1)\left(x - \sum_{j=1}^{i-1}\alpha_j(\mu)\left(\frac{1}{r(N(\mu)-j+1)} - \frac{1}{r(N(\mu)-j)}\right)\right),$$

- $i^r(\mu, x) = \max\left\{i \leq N(\mu); \alpha_i(\mu) \leq \gamma_i^r(\mu, x)\right\},$
- $\gamma^r(\mu, x) := \gamma^r_{(i^r(\mu,x)+1)\wedge 1}(\mu, x),$
- $\Phi^r(\mu, x) = \tau_{\gamma^r(\mu,x)}\mu,$

we have for any initial profile μ_0 anf for all $n \in \mathbb{N}$,

$$\mu_{n+1} = \Phi^r\left(\mu_n + \delta_{\sigma_n}, \xi_n\right). \tag{1}$$

Proof. Just after the arrival of C_n, the service profile reads $\mu := \mu_n + \delta_{\sigma_n}$. Set $T_0' := T_n$ and $\alpha_0(\mu) = 0$. For any $i \in \{1, ..., N(\mu)\}$, let T_i' be the theoretical departure of the customer $\tilde{C}_i$ whose remaining service time at T_n is $\alpha_i(\mu)$. The remaining service time of $\tilde{C}_i$ at T_{i-1}' is $\alpha_i(\mu) - \alpha_{i-1}(\mu)$, and between T_{i-1}' and T_i', $\tilde{C}_i$ is served at rate $r(N(\mu) - i + 1)$. Hence we have the induction formula

$$T_i' = T_{i-1}' + \frac{\alpha_i(\mu) - \alpha_{i-1}(\mu)}{r(N(\mu) - i + 1)},\, i \in \{1, ..., N(\mu)\}, \tag{2}$$

from which we deduce that for all $i \in \{1, ..., N(\mu)\}$,

$$T_i' = T_n + \frac{\alpha_i(\mu)}{r(N(\mu)-i+1)} + \sum_{j=1}^{i-1}\alpha_j(\mu)\left(\frac{1}{r(N(\mu)-j+1)} - \frac{1}{r(N(\mu)-j)}\right). \tag{3}$$

For any i, customer $\tilde{C}_i$ leaves the system before T_{n+1} provided that $T'_i - T_n \leq \xi_n$, which is equivalent to $\alpha_i(\mu) \leq \gamma_i^r(\mu, \xi_n)$ in view of (3). In particular, $i^r(\mu, \xi_n)$ denotes the index of the last customer leaving the system before T_{n+1} (or 0 if there is no departure between T_n and T_{n+1}). Then the system is not empty at $T_{n+1}-$ provided that $i^r(\mu, \xi_n) < N(\mu)$, and in this case, $\left\{\tilde{C}_i, i \in \{i^r(\mu, \xi_n)+1, N(\mu)\}\right\}$ is the set of customers present in the system at $T_{n+1}-$. For such $i > i^r(\mu, \xi_n)$, the remaining service time of $\tilde{C}_i$ at T_{n+1} is given by

$$\alpha_i(\mu) - \alpha_{i^r(\mu,\xi_n)}(\mu) - r(N(\mu) - i^r(\mu, \xi_n)) \left(T_{n+1} - T'_{i^r(\mu,\xi_n)}\right) = \alpha_i(\mu) - \gamma^r(\mu, \xi_n).$$

Thus the functional mapping the profile at T_n onto the profile at $T_{n+1}-$ reads

$$\Phi^r(., \xi_n) : \mu \longmapsto \sum_{i=i^r(\mu,\xi_n)+1}^{N(\mu)} \delta_{\alpha_i(\mu)-\gamma^r(\mu,\xi_n)}.$$

To obtain the announced result, remark that for any $\mu \in \mathcal{M}$ and $x \in \mathbb{R}_+^*$, for any $i < N(\mu)$ we have that

$$\gamma_{i+1}^r(\mu, x) - \gamma_i^r(\mu, x) = \frac{r\left(N(\mu) - i\right) - r\left(N(\mu) - i + 1\right)}{r\left(N(\mu) - i + 1\right)} \left(\gamma_i^r(\mu, x) - \alpha_i(\mu)\right),$$

which is nonnegative if and only if $i \leq i^r(\mu, x)$. Hence,

$$\gamma^r(\mu, x) = \max_{1 \leq i \leq N(\mu)} \gamma_i^r(\mu, x), \tag{4}$$

and in particular $\Phi^r(\mu, \xi_n) = \tau_{\gamma^r(\mu,\xi_n)}\mu$, $\mathbf{P}^0$-a.s.. □

For a fixed $x \in \mathbb{R}+$, the two following monotonicity properties of the mappings $\Phi^r(., x)$ hold, as shown in Appendix.

Lemma 3. *For any $x \in \mathbb{R}+$ and any rate function r, the mapping $\Phi^r(., x)$ is non-decreasing from $\mathcal{M}$ into itself.*

Lemma 4. *For any $x \in \mathbb{R}+$ and any $\mu \in \mathcal{M}$, for any two rate functions r and $\tilde{r}$ such that $r(i) \leq \tilde{r}(i)$ for all $i \in \mathbb{N}^*$, $\Phi^r(\mu, x) \succeq \Phi^{\tilde{r}}(\mu, x)$.*

4. The pure delay system

Let us first consider the case, where the rate function is constant with respect to the size of the system, say $r(i) = 1$ for any $i \geq 1$. This corresponds to the classical "pure delay" G/G/∞ queue: all present customers are simultaneously served at unit rate, and hence spend in the system a time equal to their service duration, which is equivalent to say that there is an infinity a servers. In this case, the recursive equation (1) driving the service profile sequence (for which a diffusion approximation is given in [5] in the M/GI/∞ case) specializes to

$$\mu_{n+1} = \tau_{\xi_n}\left(\mu_n + \delta_{\sigma_n}\right) \tag{5}$$

and a stationary service profile for the queue is a solution to the equation

$$\mu \circ \theta = \tau_\xi\left(\mu + \delta_\sigma\right). \tag{6}$$

The following lemma (see [7]) will be used in the sequel.

Lemma 5. *There exists a unique* $\mathbf{P}^0$*-a.s. finite solution to the equation*

$$L \circ \theta = [\max\{L, \sigma\} - \xi]^+, \tag{7}$$

given by

$$L := \left[\sup_{j \in \mathbb{N}^*} \left(\sigma_{-j} - \sum_{i=1}^{j} \xi_{-i}\right)\right]^+. \tag{8}$$

Proof. Existence. Loynes' Theorem for stochastic recurrences (see [6], [1]) can be applied since the mapping $x \mapsto [\max\{x, \sigma\} - \xi]^+$ is $\mathbf{P}^0$-a.s. continuous and non-decreasing. The minimal solution L to (7) classically reads as the $\mathbf{P}^0$-a.s. limit of Loynes's sequence $\left\{L_n^{[0]} \circ \theta^{-n}\right\}_{n\in\mathbb{N}}$, where $\left\{L_n^{[0]}\right\}_{n\in\mathbb{N}}$ is the initially null SRS that is defined by

$$L_{n+1}^{[0]} = \left[\max\left\{L_n^{[0]}, \sigma_n\right\} - \xi_n\right]^+ \text{ for all } n \in \mathbb{N}.$$

It is routine to check from Birkhoff's ergodic theorem (and the fact that σ is not identically zero) that L is $\mathbf{P}^0$-a.s. finite.
Uniqueness. Let $\tilde{L}$ be a solution to (7). First, remark that if $\tilde{L} > \sigma$, $\mathbf{P}^0$-a.s. would imply that on a $\mathbf{P}^0$-a.s. event,

$$\tilde{L} \circ \theta > 0 \Leftrightarrow \tilde{L} \circ \theta = \tilde{L} - \xi,$$

a contradiction to the ergodic Lemma. Hence in view of the minimality of L, we have that

$$\mathbf{P}^0\left[\tilde{L} = L\right] = \mathbf{P}^0\left[\tilde{L} \circ \theta \le L \circ \theta\right] \ge \mathbf{P}^0\left[\tilde{L} \le \sigma\right] > 0,$$

which implies that $\left\{\tilde{L} = L\right\}$ is $\mathbf{P}^0$-almost sure since it is θ-contracting. $\square$

We can now state the following result.

Theorem 1. *The equation (6) admits a finite solution, given by*

$$\mu^{PD} = \sum_{i=1}^{\infty} \delta_{\left(\sigma_{-i} - \sum_{j=1}^{i} \xi_{-j}\right)} \mathbf{1}_{\left\{\sigma_{-i} \ge \sum_{j=1}^{i} \xi_{-j}\right\}}.$$

Moreover, provided that

$$\mathbf{P}^0[L \le 0] > 0, \tag{9}$$

this solution is unique and for all ζ *such that* $Z(\zeta) \le L$, $\mathbf{P}^0$*-a.s, the sequence* $\left\{\mu_n^{[\zeta]}\right\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to* μ^{PD}.

Proof. Existence. It is a straightforward consequence of Birkhoff's ergodic theorem that

$$\mathbf{P}^0\left[\mu^{\mathrm{PD}} \in \mathcal{M}\right] = \mathbf{P}^0\left[\mathrm{Card}\left\{i \in \mathbb{N}^*, \sigma_{-i} - \sum_{j=1}^{i} \xi_{-j} \ge 0\right\} < \infty\right] > 0.$$

This θ-contracting event is thus $\mathbf{P}^0$-almost sure. On another hand, in view of Lemma 3, the mapping $\mu \mapsto \tau_\xi(\mu + \delta_\sigma)$ is $\mathbf{P}^0$-a.s. non-decreasing from $\mathcal{M}$ into itself. It is furthermore continuous for the weak topology, as easily checked from

the fact that for any $\mathcal{M}$-valued sequence $\{\nu_n\}_{n\in\mathbb{N}}$ tending weakly to ν, for any $x, s \in \mathbb{R}+$ and any $\phi \in \mathcal{C}_b$,

$$\langle \tau_x \nu_n + \delta_s, \phi\rangle = \int \phi(y-x)\, d\nu_n(y) + \phi(s) \underset{n\to\infty}{\longrightarrow} \int \phi(y-x)\, d\nu(y) = \langle \tau_x\nu + \delta_s, \phi\rangle.$$

Thus, we can follow the steps of Loynes' construction (Lemma 1), to conclude that μ^{PD} is the $\preceq$-minimal solution of (6) since it is the $\mathbf{P}^0$-a.s. limit of the sequence given for all $n \in \mathbb{N}$ by

$$\mu_n^{[\mathbf{0}]} \circ \theta^{-n} = \sum_{i=1}^{\infty} \delta_{\left(D_{-i} - \sum_{j=1}^{i} \xi_{-j}\right)} \mathbf{1}_{\left\{D_{-i} \geq \sum_{j=1}^{i} \xi_{-j}\right\}}.$$

Uniqueness. It is easily checked that for any solution μ of (6),

$$Z(\mu)\circ\theta = Z\left(\tau_\xi\left(\mu + \delta_\sigma\right)\right) = \left[Z(\mu) \vee \sigma - \xi\right]^+,$$

hence $Z(\mu) = L$, $\mathbf{P}^0$-a.s.. Moreover, since μ^{PD} is the minimal solution of (6), we have that

$$\left\{\mu = \mu^{\mathrm{PD}}\right\} \supseteq \{\mu = \mathbf{0}\} = \{Z(\mu) = 0\} = \{L = 0\}.$$

Hence, whenever (9) holds, the event $\{\mu = \mu^{\mathrm{PD}}\}$ has a positive probability. Since it is θ-invariant, it is $\mathbf{P}^0$-almost sure.

Coupling. Let ζ be a $\mathcal{M}$-valued r.v. such that $Z(\zeta) \leq L$, $\mathbf{P}^0$-a.s.. It is easy to construct another $\mathcal{M}$-valued r.v. $\tilde{\zeta}$ such that $\zeta \preceq \tilde{\zeta}$ and $Z(\tilde{\zeta}) = L$, $\mathbf{P}^0$-a.s. by setting e.g. $\tilde{\zeta} = \sum_{i=1}^{N(\zeta)-1} \delta_i(\zeta) + \delta_L$. From Lemma 3, it follows by induction that $\mu_n^{[\zeta]} \preceq \mu_n^{[\tilde{\zeta}]}$, $\mathbf{P}^0$-a.s. for all $n\in\mathbb{N}$. Remark now that for all $n\in\mathbb{N}$, $Z\left(\mu_n^{[\tilde{\zeta}]}\right) = L\circ\theta^n$, as easily checked by induction. Hence, for all $n \in \mathbb{N}$, we have

$$\mathcal{E}_n := \{L\circ\theta^n = 0\} = \left\{Z\left(\mu_n^{[\tilde{\zeta}]}\right) = 0\right\} = \left\{\mu_n^{[\tilde{\zeta}]} = \mathbf{0}\right\} \subseteq \left\{\mu_n^{[\zeta]} = \mathbf{0}\right\}.$$

Therefore, $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a stationary sequence of renovating events of length 1 for $\left\{\mu_n^{[\zeta]}\right\}_{n\in\mathbb{N}}$ (see [3, 4]) for any ζ such that $Z(\zeta) \leq L$, $\mathbf{P}^0$-a.s.. Assumptions (9) implies the coupling property for such an initial condition in view of Corollary 2.5.1 of [1]. $\square$

As simple consequences of the latter result, let us remark the following coupling properties.

Corollary 1. *Under condition (9), for any ζ such that $Z(\zeta) \leq L$, $\mathbf{P}^0$-a.s,*

(i) $\left\{X_n^{[N(\zeta)]}\right\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to* $N\left(\mu^{PD}\right)$;

(ii) $\left\{W_n^{[\langle\zeta, I\rangle]}\right\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to* $\langle\mu^{PD}, I\rangle$.

5. Processor Sharing queues

We shall now consider the case, where the rate function depends on the number of customers in the system at current time. We assume hereafter that the non-decreasing function r is such that

$$\sup_{n\in\mathbb{N}^*} n.r(n) \leq 1, \tag{10}$$

$$K_r = \inf_{n\in\mathbb{N}^*} n.r(n) > 0. \tag{11}$$

Assumption (10) amounts to say that there is a single server, in that the throughput at time t, given by $Q(t).r(Q(t))$, may not exceed one. A typical case is the classical Processor Sharing queue: assume that $r(n) = n^{-1}$ for any n (and hence $K_r = 1$), meaning that all customers are served at a rate that is inversely proportional to the number of customers. In that case the server works at unit rate whatever the number of customers in the system. Whenever $K_r < 1$, the number of customers affects the velocity of service, so that the total throughput may be less than one. We assume nevertheless in (11) that a minimal throughput K_r is granted for a given r, *i.e.* the server always achieves at least K_r unit of work per unit of time. An example is provided by the following idealistic scenario: the server works at unit rate whenever there is only one customer in the system ($r(1) = 1$), and the interferences (or operating cost) when there are several customers in service at the same time decreases by half the efficiency of the server, so that $r(i) = 1/(2i)$ for any $i \geq 2$, which implies in particular that (11) is met for $K_r = 1/2$.

In view of Lemma 2, a stationary service profile is a solution to the equation

$$\mu \circ \theta = \Phi^r \left(\mu + \delta_\sigma, \xi\right). \tag{12}$$

We have the following result.

Theorem 2. *Let r be a rate function satisfying assumptions (10) and (11). Then provided that*

$$\mathbf{E}^0 \left[\sigma\right] < K_r \mathbf{E}^0 \left[\xi\right], \tag{13}$$

the equation (12) admits a unique finite solution μ^r. Moreover, for any $\mathcal{M}$-valued r.v. ζ such that $\langle \zeta, I\rangle \leq W^{K_r}$, $\mathbf{P}^0$-a.s. (where W^{K_r} is the unique solution of (14)), the sequence $\left\{\mu_n^{[\zeta]}\right\}_{n\in\mathbb{N}}$ converges with strong backward coupling to μ^r.

Proof. Existence. Fix r satisfying (10) and (11). From Loynes's fundamental stability result, the equation

$$W \circ \theta = \left[W + \sigma - K_r \xi\right]^+ \tag{14}$$

admits a unique $\mathbf{P}^0$-a.s. finite solution, say W^{K_r}, provided that (13) holds. Let $\tilde{r}$ be the rate function such that for all $\mu \in \mathcal{M}$, $\tilde{r}(\mu) = K_r/N(\mu)$, so that the throughput under $\tilde{r}$ always equals K_r whenever the system is non-empty. Let ζ be a $\mathcal{M}$-valued r.v. such that $\langle \zeta, I\rangle \leq W^{K_r}$ and

$$\tilde{\zeta} = \zeta + \delta_{W^{K_r} - \langle\zeta, I\rangle} \mathbf{1}_{W^{K_r} > \langle \zeta, I\rangle}.$$

Is then clear that $\langle\tilde{\zeta}, I\rangle = W^{K_r}$. Moreover, we have $\mathbf{P}^0$-a.s. for all $n \in \mathbb{N}$

$$\langle \mu_{n+1}^{\tilde{r},[\tilde{\zeta}]}, I\rangle = \left[\langle \mu_n^{\tilde{r},[\tilde{\zeta}]}, I\rangle + \sigma_n - K_r \xi_n\right]^+,$$

as the throughput equals K_r at any time (as easily checked from Lemma 2), so that $\langle \mu_{n+1}^{\tilde{r},[\tilde{\zeta}]}, I\rangle = W^{K_r} \circ \theta^n$ for all $n \in \mathbb{N}$. On another hand, $\zeta \preceq \tilde{\zeta}$, hence in view of Lemmas 3 and 4, an immediate induction shows that $\mu_n^{r,[\zeta]} \preceq \mu_n^{\tilde{r},[\tilde{\zeta}]}$ for all $n \in \mathbb{N}$, which implies in turn that

$$\langle \mu_n^{r,[\zeta]}, I\rangle \leq \langle \mu_n^{\tilde{r},[\tilde{\zeta}]}, I\rangle = W^{K_r} \circ \theta^n \text{ for all } n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$, on $\mathfrak{A}_n := \left\{W^{K_r} \circ \theta^n = 0\right\}$, we have that $\langle \mu_n^{r,[\zeta]}, I\rangle = 0$, hence $\mu_n^{r,[\zeta]} = \mathbf{0}$ and

$$\mu_{n+1}^{r,[\zeta]} = \Phi^r \left(\delta_{\sigma_n}, \xi_n\right).$$

Therefore $\left\{\mu_n^{r,[\zeta]}\right\}_{n\in\mathbb{N}}$ admits $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ as a stationary sequence of renovating events of length 1. Furthermore, the event $\mathfrak{A}_0=\left\{W^{K_r}=0\right\}$ has a striclty positive probability since the contrary would imply that

$$\mathbf{E}^0\left[W^{K_r}\circ\theta-W^{K_r}\right]=\mathbf{E}^0\left[\sigma-K_r\xi\right]<0,$$

an absurdity in view of the ergodic Lemma. Then it follows from [1], Th. 2.5.3, that there is strong backwards coupling of $\mu_n^{r,[\zeta]}$ with the stationary sequence $\{\mu^r\circ\theta^n\}_{n\in\mathbb{N}}$, where μ^r is a proper solution to (12).

Uniqueness. Fix r and $\tilde{r}$ be as above. There exists a solution $\mu^{\tilde{r}}$ to (12). Then, we have $\mathbf{P}^0$-a.s.

$$\langle\mu^{\tilde{r}},I\rangle\circ\theta=\left\langle\Phi^{\tilde{r}}\left(\mu^{\tilde{r}}+\delta_\sigma,\xi\right),I\right\rangle=\left[\langle\mu^{\tilde{r}},I\rangle+\sigma-K_r\xi\right]^+,$$

hence $\langle\mu^{\tilde{r}},I\rangle$ equals W^{K_r}, $\mathbf{P}^0$-a.s.. Moreover, on $\left\{\langle\mu^r,I\rangle\le W^{K^r}\right\}$, we have in view of Lemma 2 that

$$\langle\mu^r,I\rangle\circ\theta\le\left\langle\Phi^{\tilde{r}}\left(\mu^r+\delta_\sigma,\xi\right),I\right\rangle=\left[\langle\mu^r,I\rangle+\sigma-K_r\xi\right]^+\le W^{K_r}\circ\theta,\ \mathbf{P}^0-\text{a.s.},$$

thus the event $\left\{\langle\mu^r,I\rangle\le W^{K_r}\right\}$ is θ-contracting. Moreover,

$$\mathbf{P}^0\left[\langle\mu^r,I\rangle\le W^{K_r}\right]\ge\mathbf{P}^0\left[\langle\mu^r,I\rangle=0\right]>0,$$

as another consequence of (13) and the ergodic Lemma. Therefore, $\langle\mu^r,I\rangle\le W^{K_r}$, $\mathbf{P}^0$-a.s., so that

$$\mathfrak{A}_n\subseteq\{\langle\mu^r,I\rangle\circ\theta^n=0\}=\{\mu^r\circ\theta^n=\mathbf{0}\}.$$

Consequently, $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ is a stationary sequence of renovating events of length 1 for $\{\mu^r\circ\theta^n\}_{n\in\mathbb{N}}$ for any solution μ^r of the equation (12) associated to the rate r. Since $\mathbf{P}^0\left[\mathfrak{A}_0\right]$, there exists a unique solution to (12) in view of Remark 2.5.3. in [1]. □

We have in particular:

Corollary 2. *Under condition (13), for any ζ such that $\langle\zeta,I\rangle\le W^{K_r}$, $\mathbf{P}^0$-a.s.,*

(i) $\left\{X_n^{[N(\zeta)]}\right\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to* $N\left(\mu^r\right)$;

(ii) $\left\{W_n^{[\langle\zeta,I\rangle]}\right\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to* $\langle\mu^r,I\rangle$.

Appendix A. Proofs of monotonicity

For easy checking, we present hereafter the details of the derivations proving Lemmas 3 and 4.

Proof of Lemma 3. We fix again $x\in\mathbb{R}+$ and $\mu,\,\nu\in\mathcal{M}$ such that $\mu\preceq\nu$. Whenever $i^r(\mu,x)<N(\mu)$ (otherwise $\Phi^r(\mu,x)=\mathbf{0}$), we have that

$$\sum_{j=1}^{N(\nu)-N(\mu)+i^r(\mu,x)}\alpha_j(\nu)\left(\frac{1}{r(N(\nu)-j+1)}-\frac{1}{r(N(\nu)-j)}\right)$$
$$\ge\sum_{j=1}^{i^r(\mu,x)}\alpha_j(\mu)\left(\frac{1}{r(N(\mu)-j+1)}-\frac{1}{r(N(\mu)-j)}\right),$$

which implies that

$$\begin{aligned}\alpha_{N(\nu)-N(\mu)+i^r(\mu,x)+1}(\nu) &\geq \alpha_{i^r(\mu,x)+1}(\mu)\\ &\geq r(N(\mu)-i^r(\mu,x))\left(x-\sum_{j=1}^{i^r(\mu,x)}\alpha_j(\mu)\left(\frac{1}{r(N(\mu)-j+1)}-\frac{1}{r(N(\mu)-j)}\right)\right)\\ &\geq \gamma^r_{N(\nu)-N(\mu)+i^r(\mu,x)+1}(\nu,x).\end{aligned}$$

This means that $i_0(\nu,x)\leq N(\nu)-N(\mu)+i_0(\mu,x)$, i.e. $N\left(\Phi^r(\mu,x)\right)\leq N\left(\Phi^r(\nu,x)\right).$ Hence in view of (4), we have

$$\begin{aligned}\gamma(\mu,\xi) = \gamma^r_{i^r(\mu,\xi)+1}(\mu,x) &\geq \gamma^r_{(i^r(\nu,\xi)+N(\mu)-N(\nu))^++1}(\mu,x)\\ &\geq r\left(N(\nu)-i^r(\nu,x)\right)\left(x-\sum_{j=1}^{i^r(\nu,x)}\alpha_j(\nu)\left(\frac{1}{r\left(N(\nu)-j+1\right)}-\frac{1}{r\left(N(\nu)-j\right)}\right)\right)\\ &= \gamma^r(\nu,x),\end{aligned}$$

which clearly implies that $\Phi^r(\mu,x)\preceq\Phi^r(\nu,x)$. □

Proof of Lemma 4. We now fix $\mu\in\mathcal{M}$ and $x\in\mathbb{R}+$. For any two rate functions r and $\tilde{r}$ such that $r(i)\leq\tilde{r}(i)$ for any $i\in\mathbb{N}^*$, the induction formula (2) straightforwardly shows that $i^r(\mu,x)\geq i^{\tilde{r}}(\mu,x)$ i.e. $N\left(\Phi^r(\mu,x)\right)\leq N\left(\Phi^{\tilde{r}}(\mu,x)\right)$. Hence, as in the previous proof, $\gamma^r(\mu,x)\leq\gamma^{\tilde{r}}(\mu,x)$. □

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